

**Externally Driven Nonlinear Oscillator, Painlevé Test, First Integrals and Lie Symmetries**

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For arbitrary constants  $c_1, c_2$  and an arbitrary smooth functions  $f$  the driven anharmonic oscillator  $d^2u/dt^2 + c_1 du/dt + c_2 u + u^3 = f(t)$  cannot be solved in closed form. We apply the Painlevé test to obtain the constraint on the constants  $c_1, c_2$  and the function  $f$  for which the equation passes the test. We also give the Lie symmetry vector field and first integrals for this equation.

For arbitrary constants  $c_1, c_2$  and an arbitrary smooth function  $f$  the anharmonic oscillator

$$\frac{d^2u}{dt^2} + c_1 \frac{du}{dt} + c_2 u + u^3 = f(t) \tag{1}$$

cannot be solved in closed form.

We apply the Painlevé test [1, 2, 3] to obtain the constraint on the constants  $c_1, c_2$  and the function  $f$  for which (1) passes the test. The constraint on  $c_1$  and  $c_2$  gives an algebraic equation and the constraint of  $f$  is a linear differential equation. We solve these equations and give the Lie point symmetry and the first integral for this special case of (1).

Let us first discuss the Painlevé test for (1). A remark is in order for applying the Painlevé test for non-autonomous systems. The coefficients that depend on the independent variable must themselves be expanded in terms of  $t - t_1$ , where  $t_1$  is the pole position and we use the identity  $t \equiv (t - t_1) + t_1$ . If non-autonomous terms enter the equation at lower order than the dominant balance the above mentioned expansion turns out to be unnecessary whereas if the nonautonomous terms are at dominant balance level they must be expanded with respect to  $t - t_1$ . We assume that  $f$  does not enter the expansion at dominant level.

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Before we study (1) we give a brief review of the special case

$$\frac{d^2u}{dt^2} + c_1 \frac{du}{dt} + c_2 u + u^3 = 0, \tag{2}$$

where  $c_1$  and  $c_2$  are constants. Equation (2) is considered in the complex domain with  $c_1$  and  $c_2$  real. For the sake of simplicity we do not change the notation. Inserting the Laurent expansion

$$u(t) = \sum_{j=0}^{\infty} a_j (t - t_1)^{j-n}, \tag{3}$$

where  $t_1$  denotes the pole position, yields  $n=1$  and  $a_0^2 = -2$ . The expansion coefficients  $a_1, a_2$ , and  $a_3$  are determined by

$$\begin{aligned} 3a_1 a_0 &= c_1, & 3a_2 a_0 &= -c_2 - 3a_1^2, \\ 4a_3 &= c_1 a_2 + c_2 a_1 + a_1^3 + 6a_0 a_1 a_2. \end{aligned} \tag{4}$$

The expansion coefficient  $a_4$  is arbitrary in expansion (3) if

$$c_1^2 (2c_1^2 - 9c_2) = 0. \tag{5}$$

This means  $r = 4$  is a so-called resonance (compare [1, 2, 3] and references therein). The solution  $c_1 = 0$  is the trivial case. To summarize: If  $2c_1^2 = 9c_2$ , then the general solution of (2) can be expressed in terms of Jacobi elliptic functions. For this case (i.e.  $2c_1^2 = 9c_2$ ) we can find an explicitly time-dependent first integral, namely

$$I(t, u, \dot{u}) = \exp\left(\frac{4}{3} c_1 t\right) \left( \left( \dot{u} + \frac{c_1 u}{3} \right)^2 + \frac{1}{2} u^4 \right). \tag{6}$$

If condition (5) is satisfied, then (2) admits two Lie symmetry vector fields

$$Z_1 = \frac{\partial}{\partial t}, \quad Z_2 = -\frac{c_1}{3} \exp\left(\frac{c_1 t}{3}\right) u \frac{\partial}{\partial u} + \exp\left(\frac{c_1 t}{3}\right) \frac{\partial}{\partial t}. \tag{7}$$

Let us now consider (1). Inserting the ansatz

$$u(t) = \sum_{j=0}^{\infty} u_j(t) \phi(t)^{j-n} \tag{8}$$

with  $n=1$  into (1), we find at the resonance  $r=4$  the condition

$$-27\sqrt{-2} \frac{df}{dt} - 27\sqrt{-2} c_1 f - 18c_2 c_1^2 + 4c_1^4 = 0. \tag{9}$$

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Since  $\sqrt{-2}$  is imaginary and  $c_1$  and  $c_2$  are real, it follows that (9) decomposes into two equations, namely

$$\frac{df}{dt} + c_1 f = 0 \quad (10)$$

and condition (5). The general solution of (10) is given by

$$f(t) = C e^{-c_1 t}. \quad (11)$$

Consequently,

$$\frac{d^2 u}{dt^2} + c_1 \frac{du}{dt} + \frac{2}{9} c_1^2 u + u^3 = C \exp(-c_1 t) \quad (12)$$

passes the Painlevé test. Equation (12) admits one Lie symmetry vector field, namely  $Z_2$  given by (7). Now

(12) can be derived from a Lagrangian function

$$\mathcal{L}(u, \dot{u}, t) = e^{c_1 t} \left( \frac{1}{2} \dot{u}^2 - V(u, t) \right), \quad (13)$$

where

$$V(u, t) = \frac{1}{9} c_1^2 u^2 + \frac{1}{4} u^4 - C u e^{-c_1 t}. \quad (14)$$

Thus we can apply Noether's theorem to find a first integral from the Lie symmetry vector field  $Z_2$ . We obtain

$$I(t, u, \dot{u}) \quad (15) \\ = \exp\left(\frac{4}{3} c_1 t\right) \left( \left( \dot{u} + \frac{c_1 u}{3} \right)^2 + \frac{1}{2} u^4 - 2 C u e^{-c_1 t} \right).$$

We used REDUCE [4] and C++ [5] for most of the calculations performed in this paper.

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